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Solution of Advection Dispersion Equation Along with Boundary Value Using Differential Transform Method

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Abstract Groundwater contamination is a major threat to the society, mostly when it occurs from non-point type of sources. Easiest way to describe the non-point source is to model the phenomenon mathematically. In this study, an effective computational method is developed to solve the boundary value problem. Finding the initial condition for a problem may seem easy at first, but often the real challenge lies in determining the appropriate boundary conditions and solving the associated boundary value problems. The flow of fluids through a porous medium has been successfully modeled using the advection dispersion equation (ADE) while considering the appropriate boundary conditions. To solve this complex problem, a powerful technique known as the Differential Transform Method (DTM) has been employed. The results of this study show that the differential transform method can achieve a unique result.

1. Introduction:

In mathematics, a differential equation together with a set of additional constraints i.e. boundary conditions are called boundary value problems. We may solve the boundary value problems by finding the solution of differential equation which satisfies the boundary conditions. The Boundary Value Problems recently are often discussed in the areas of fluid dynamics, hydrodynamics, hydromagnetic Stability, theory of thermal explosions, inelastic flows, boundary layer theory, the field of astrophysics, the theory of elastic stability and various applied sciences. Generally, it is very difficult to solve analytically the boundary value problems. To overcome the problem, many researchers have proposed various numerical techniques. In 1978, Inokuti et al. introduced the variational iteration method. This method is a modified version of the general Lagrange multiplier method, specifically designed to solve non-linear initial and boundary value problems. R. P. Agarwal (1986) solved the boundary value problem by using higher order differential equations. The eighth order boundary value problem was solved by Siddiqi et al. (1996). These problems were pointed out by Wazwas (2000) where the modified decomposition method was used. Caglar et al. (2006) solved the fifth order boundary value problem by B-spline interpolation. Mohammad-Jawad (2010) successfully solved the eighth-order boundary value problem. Mohyud-Din & Yildirim, A. (2010) also studied the higher order boundary value problem by modified variational iteration method. Siddiqi et al. (2012) solved seventh order Boundary Problems that arise in modeling induction motors with two rotor circuits. The comparison of the results obtained from B-spline interpolation of the fifth order boundary value problem with those obtained from the finite element and finite volume methods, as presented by Triphathi in 2015, revealed interesting insights. Opanuga et al. (2015) describe the numerical solution of a two-point boundary value problem by differential transform method. Xie, L. J., Zhou, C. L., & Xu, S. (2016) gives the idea special type of boundary value problems by using improved differential transform method. The main disadvantage of these methods is the occurrence of undesirable numerical errors. These methods may produce oscillatory outcomes. To avoid such oscillation of results, the Differential Transform Method was introduced in this study. A number of linear and non-linear problems can be solved easily and efficiently by this method. To illustrate the efficiency and implementation of the method numerical examples have been given.

Differential Transform Methods are widely used in various mathematical models. This method provides a solution with easily computable components. J. k. Zhou (1986) first proposed the concept of Differential Transform Method which are numerical methods and are easier to deal with. He applied to solve the boundary value problem of the ninth and twelfth orders. The initial value problems of all coordinates can be solved by the Differential Transform Method. In this paper, we have employed the Differential Transform Method to effectively solve higher order Boundary Value Problems and obtain a distinct and conclusive solution

2. Mathematical modelling and Solution using DTM:

In this present work authors considered fluid flow through porous medium and modelled the problem as follows:

$$\frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial x^2} - u_0 \frac{\partial c}{\partial x} \tag{1}$$

Along with the initial condition

$$c(x,0) = c_0 \tag{2}$$

The boundary conditions are considered as follows

$$c(0,t) = ae^{-t} \tag{3}$$

$$\frac{\partial c}{\partial x} = 0 \; when \; x \to \infty \tag{4}$$

So, in this present problem, the aquifer is initially contaminated with a constant amount of pollutants. The current issue at hand pertains to a one-dimensional problem. Here, the source condition manifests itself in one portion of the boundary, while a no-flux condition exists in the semi-infinite part of said boundary.

This present problem is solved using the Differential transform method. DFM is primarily utilized for solving initial value problems. We can consider the concentration as an unknown function, which guarantees that it satisfies the boundary conditions in a flexible way. By doing so, we ensure no loss of generality. The value of the concentration is written as follows:

$$c(x,t) = uxe^{-x} + ae^{-t}e^{-x}, \tag{5}$$

where $u(x,t)$ is the new concentration function considered in this present manuscript. In corporation of boundary condition enables us to solve the problem using DTM.

Authors used equation (5), to modify (1) and (2), modified equations can be written as follows:

$$\begin{aligned} xe^{-x}\frac{\partial u}{\partial t} - ae^{-t}e^{-x} \qquad &= D_0[xe^{-x}\frac{\partial^2 u}{\partial x^2} + e^{-x}\frac{\partial u}{\partial x} - xe^{-x}\frac{\partial u}{\partial x} + e^{-x}\frac{\partial u}{\partial x} - ue^{-x} - xe^{-x}\frac{\partial u}{\partial x} - ue^{-x} + uxe^{-x} + ae^{-t}e^{-x}] \\ &\quad -u_0[xe^{-x}\frac{\partial u}{\partial x} + ue^{-x} - uxe^{-x} - ae^{-t}e^{-x}] \end{aligned} \tag{6}$$

$$u(x,0) = \frac{c_0 - ae^{-x}}{x} e^{x} \tag{7}$$

Now solving (6) and (7) using DTM we get the solution as follows:

$$\begin{aligned} U(x,t) &= U(0,0) + U(1,0)x + U(0,1)t + U(1,1)xt + U(1,2)xt^2 + U(2,1)x^2t + U(2,0)x^2 \\ &= 1 + (e-1)x + ((2e-2-2+a)D_0 + (a-1)u_0 + a)t + (D_0 + u_0 - 4D_0(e-1) - 2u_0(e-1))xt \\ &\quad + (\frac{t}{2}[-(4D_0 + 2u_0)\{(5-4e)D_0 + (3-2e_0)u_0\} + (D_0 + u_0)\frac{1}{x}\{(2e-4+a)D_0 + (a-1)u_0 + a\}])xt^2 \\ &\quad + (x[(e-1)D_0 + (e-1)u_0])x^2t + \frac{c_0e^2 - a}{2}x^2 \end{aligned} \tag{8}$$

Authors achieve the complete solution by putting (8) in (5).

3. Result and Discussion:

The solution of the advection-dispersion equation always gives us a clear idea of the concentration profile of contaminants as they move through the porous medium. It is often required to understand how the contaminant concentration shows the impact along with the increasing time. If the concentration of contaminants decreases over time or distance, it means that the solute will be naturally removed from the fluid, resulting in minimal impact on human health.

In Fig. 1, authors depict the contaminant concentration profile with time. It is clear from the figure that the concentration is increasing with time. In this current figure, the authors have taken into account the time in days. It is clear that initially, there is a gradual increase in the concentration rate, which continues up to 60 days. However, beyond this point, there is a sudden and significant surge in the concentration profile, surpassing the previous 60 days. Actually, time dependence of the solution as well as initial boundary conditions and groundwater flow parameters shows the impact. Initially, the function has a lower value, but as time increases, the concentration value also increases, leading to this type of situation.

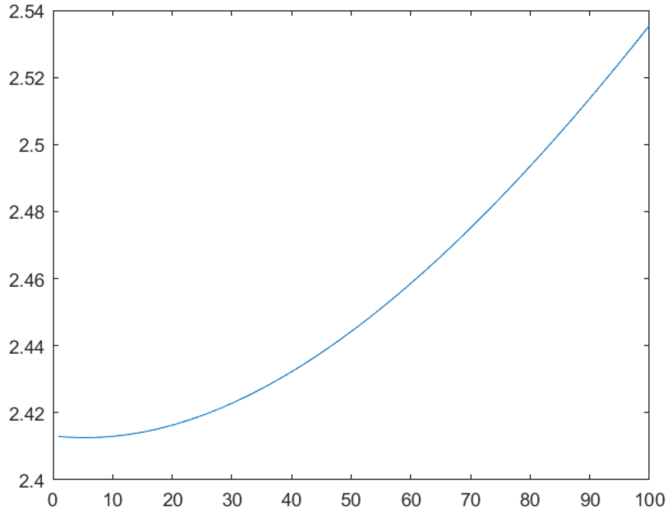

Figure 1: Concentration with time (time is in days)

The concentration profile typically decreases as the distance increases. In this solution, it is evident that the concentration increases gradually with distance. By plotting the data at a distance of 200 meters from the source, we can observe that the concentration level has risen to approximately 4.5 units, a significant increase from the initial level of 0.5 units. Looking at these trends one can say that after 1 km the contaminant concentration will decrease.

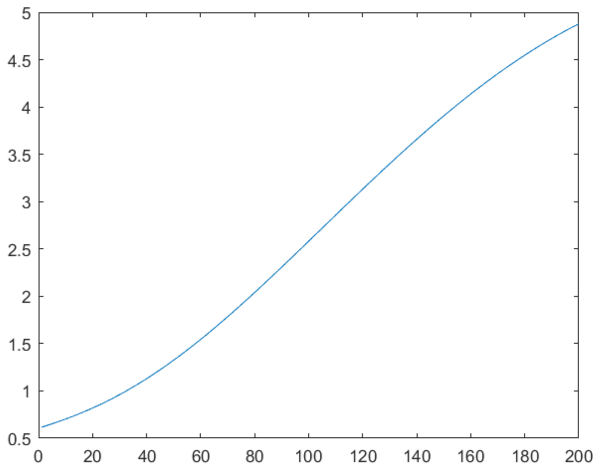

Figure 2: Concentration with distance (distance in meters)

In this particular research, we employ the DTM (Differential Transform Method), which effectively tackles initial value problems. Our main goal, however, is to broaden the horizons of this method by utilizing it to effectively solve boundary value problems as well. The obtained result is quite similar with the previously published work in the field of contaminant transport problem.

4. Conclusions

Solving this present problem authors can conclude the following:
1. Contaminant concentration increases with increasing time and may show its adverse impact on human health.
2. The level of contaminant concentration increases with increasing distance and after a certain distance the level of contamination started decreasing.
3. DTM can handle the boundary value problem and provides more or less good results.
4. Present problems can later serve as benchmarks for semi-analytical solutions.

Data accessibility. This article has no additional data.
Conflict of interest declaration. We declare we have no competing interests.
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